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ON THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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ABSTRACT

In this paper, we combine the homotopy perturbation method, Sumudu transform and He's polynomials to obtain the approximate /exact solution of some partial differential equations of fractional order. The fractional derivative is considered in Caputo sense. Some illustrate examples are presented to show the accuracy and easy implementation of this method.

Keywords: fractional calculus, homotopyperturbation method, Sumudutransform, Fractional partial differential equations.

الملخص

في هذا البحث دمجنا طريقة التشويش المضطرب معتحويل صموديوو وكثيرة الحدود هي للحصول على الحل التقريبي أو الصحيح لبعض المعادلات التفاضليةالجزئية ذات الرتبة الكسرية. التفاضل الكسري المستخدم في صيغة كابوتو. تم إعطاء بعض الأمثلة لتوضيح دقة وشهولة تنفيذ هذه الطريقة.

الكلمات المفتاحية :الحساب الكسري، طريقة التشويش المضطرب، تحويل صموديو، المعادلات التفاضليةالجزئية الكسرية

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1. INTRODUCTION

The traditional partial differential equations may not be adequate for describing the underlying phenomena for example, transport phenomena in complex systems such as random fractal structures, which exhibit many anomalous features that are qualitatively different from the standard behavior characteristics of regular systems[5].In recent years considerable interest in fractional partial differential equations(FDEs) has been simulated by applications that it finds in numerical analysis and in the different areas of physical chemical processand engineering including fractal phenomena [3,6,16,17]. There is a growing need to find the solution of these equations. However, most of these equation are difficult or impossible to solve analytically. As a consequence, an effective and easy-to-use numerical and approximate methods are needed. Over the last decades several analytical/approximate methods have been developed to solve FPDEs, some examples of these methods are homotopy analysis method [11,20],Adomain decomposition method [8,9],Laplace transform method [2,23],Variational iteration method [10],Homotopy perturbation method [12,21], Sumudu transform method [1,4,13,14]. Furtherhomotopy perturbation methods are combinedwith Laplace transform to solvemanyproblemssuch as one dimensionalnonhomogeneous partial differentialequationswith a variable coefficient [26], and itcombinedwithSumudutransformfor getting the analytical solution of the fractional Black-Scholes option pricingequation[27].

This paper extend Sumudu transform coupled to homotopy perturbation method to derivation of exact solution of FPDEs where the method based on convergent series.
The paper is organized as follows: In section 2 we give some definition in fractional calculus. In section 3 we present the basic idea of homotopy perturbation method. In section4 we extend the homotopy perturbation Sumudu transform method (HPSTM) for fractional partial differential equations. In section 5 we apply (HPSTM) to solve various types of FPDEs. Finally, In section 6 we summarize our work in conclusion

2. PRELIMINARIES AND NOTATIONS

Definition 2.1:[15]
A real function $f(x), x > 0$is said to be in space $C_\mu, \mu \in R$ if there exists a real number $p \geq \mu$,such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C(0,\infty)$ and it is said to be in the space C_μ^n if and only if $f^n \in C_\mu, n \in N$
Definition 2.2:[18]
The Riemann-Liouville fractional integral operator of order $\alpha > 0$is defined as:

$$(J^n f) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \quad (x > a, \alpha > 0) \tag{2.1}$$

Definition 2.3:[19]
The Caputo fractional derivative of a function $f(t)$ of order $\alpha$is defined as:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f^n(t)dt}{(x-t)^{\alpha+1-n}}, \quad n-1 < \alpha \leq n \tag{2.2}$$

Definition 2.4:[22]
Consider a set A defined as

$$A = \left\{ f(t) \colon \exists\, M, \tau_1, \tau_2 > 0, |f(t)| \leq M e^{\frac{|t|}{\tau_j}}\, if\ t \in (-1)^j \times [0,\infty) \right\}. \tag{2.3}$$

For all real $t \geq 0$, the Sumudu transform of a function $f(t) \in A$, denoted by $S[f(t)] = F(u)$, is defined as

$$S[f(t)](u) = F(u) = \int_0^\infty e^{-t} f(ut)dt, \quad u \in (-\tau_1, \tau_2) \tag{2.4}$$

The function $f(t)$ in equation (1) is called the inverse Sumudu transform of $F(u)$ and is denoted by $f(t) = S^{-1}[F(u)]$

Proposition 2.1: Let $M(u), N(u)$ be Sumudu of $f(t)$, and $g(t)$, respectively, then the Sumudu of convolution

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \tag{2.5}$$

is given by,

$$S[(f * g)(t)] = uM(u)N(u) \tag{2.6}$$

In particular, if $g(t) \cong 1$, then the Sumudu of anti-derivative of the function, if $f(t)$, is

$$S[(f * 1)(t)] = S[\int_0^t f(\tau)\, g(t-\tau)\, d\tau\,] = uM(u) \tag{2.7}$$

Theorem 2.1: [7]

Let $f(t) \in A$, let F_m^* denote the Sumudu transform of derivative, $f^m(t)$of $f(t)$, then for $m \geq 1$

$$F_m^* = [F^m(t); u] = u^{-m}\left[F(u) - \sum_{k=0}^{m-1} u^k f^k\right], \quad m \geq 1. \tag{2.8}$$

The Sumudu of fractional derivative can be obtained by generalization theorem (2.1) and using proposition (2.1).

Theorem 2.2: [30]

Let $m \in N$, $m-1 < \alpha \leq m$, and $F(u)$ is the Sumudu of a function $f(t)$, then the Sumudu $F_\alpha^C(u)$, of Caputo fractional derivative of $f(t)$ is given by

$$F_\alpha^C(u) = S[D_t^\alpha f(t)] = u^{-\alpha}\left[F(u) - \sum_{k=1}^{m} u^{k-1} f^{k-1}(t)\right]_{t=0} \tag{2.9}$$

For the properties of Sumudu transform and its derivatives see [28,29].

3. HOMOTOPY PERTURBATION METHOD

The homotopy perturbation method was first proposed by [12] is applied to various problems. To illustrate the basic idea of this method, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0 \quad r \in \Omega \tag{3.1}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = g(x,y,t), \quad r \in \Gamma \tag{3.2}$$

Subject to initial condition:

$$u^{(k)}(0) = c_k \tag{3.3}$$

where$A$ is a general differential operator, B is boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of domain Ω.

In general, the operator A can be divided into two parts L and N where L is a linear operator while N is the nonlinear operator. Eq.(3.1) therefore can be written as follows:

$$L(u) + n(u) - f(r) = 0$$

By thehomotopy technique [24,25] we construct a homotopy $v(r,p): \Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \quad p \in [0,1], r \in \Omega \tag{3.4}$$

Or

$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N\ (v) - f(r)] = 0$ (3.5)

where$p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq (3.1) which satisfies the boundary conditions.

From (3.4) and (3.5) we have

$H(v,0) = L(v) - L(u_0) = 0$ (3.6)

$H(v,1) = A(v) - f(r) = 0$ (3.7)

The changing in the process of p from zero to unity is just that of $H(r,p)$ from $u_0(r)$ to $u(r)$.In topology this is called deformation and $L(v) - L(u_0)$, and $A(v) - f(r)$ are called homotopic.

Now, assume that the solution of equation (3.4) and (3.5) can be expressed as

$v = v_0 + pv_1 + p^2 v_2 + \cdots\cdots$

The approximate solution of Eq. (3.1) can be obtained by setting $p = 1$

$$u(x,t) = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots\cdots = \sum_{n=0}^{\infty} v_n$$

4. THE IDEA OF HOMOTOPY PERTURBATION COUPLED WITH SUMUDU TRANSFORM METHOD

To illustrate of the basic idea of this method, we consider the following nonlinear fractional differential equation

$$D_t^\alpha u(x,t) + L[x]u(x,t) + N[x]u(x,t) = q(x,t), \quad m-1 < \alpha \le m \qquad (4.1)$$

with initial conditions

$$\frac{\partial^k u(x,0)}{\partial t^k} = h_k(x), \qquad k = 1,2,3,\cdots\cdots, m-1 \qquad (4.2)$$

where$D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ is a fractional Caputo derivative of function $u(x,t)$, L is the linear differential operator, N is the nonlinear differential operator and $q(x,t)$ is the source term.

Now, we applying Sumudu transform on both sides of (4.1)

$$S[D_t^\alpha u(x,t)] + S[L[x]u(x,t) + N[x]u(x,t)] = S[q(x,t)]$$

using the differential property of Sumudu transform we have

$$S[u(x,t)] = f(x) - u^\alpha\big[S[L[x]u(x,t) + N[x]u(x,t)]\big] = u^\alpha\big[S[q(x,t)]\big] \qquad (4.2)$$

Operating with Sumudu invers on both sides of (4.2)

$$u(x,t) = Q(x,t) - S^{-1}\big\{u^\alpha\big[S[L[x]u(x,t) + N[x]u(x,t)]\big]\big\} \qquad (4.3)$$

where$Q(x,t) = \left[\sum_{k=0}^{m-1} w^k \frac{\partial^k u(x,0)}{\partial t^k}\right] + S^{-1}\big\{u^\alpha\big[S[q(x,t)]\big]\big\}$represents the termarisingfrom the source term and the prescribed initial conditions.

Now applying the classical HPM where the solution can be expressed as a power series in p as given below

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n\,(x,t) \tag{4.4}$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$).

We can decomposed the nonlinear term as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \tag{4.5}$$

whereH_n are He's polynomials of $u_0, u_1, u_2, \cdots\cdots, u_n$, and it can be calculated by the following formula

$$H(u_0, u_1, u_2, \cdots\cdots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, n = 0,1,2, \cdots\cdots$$

By substituting Eq. (4.4)and(4.5) in (4.3) and using HPM we get

$$\sum_{n=1}^{\infty} p^n u_n\,(x,t) = Q(x,t) + p\big(-S^{-1}\{u^{\alpha}\big[S[L[x]u(x,t) + N[x]u(x,t)]\big]\}\big).$$

This is the coupling of Sumudu transform and homotopy perturbation method using H's polynomials. By equating the coefficient of corresponding power of p on both sides, the following approximations are obtained as

p^0: $u_0\,(x,t) = Q(x,t)$

p^1: $u_1(x,t) = -\big(S^{-1}\{u^{\alpha}\big[S[L[x]u_0(x,t) + H_0(u)]\big]\}\big)$

p^2: $u_2(x,t) = -\big(S^{-1}\{u^{\alpha}\big[S[L[x]u_1(x,t) + H_1(u)]\big]\}\big)$

p^3: $u_3(x,t) = -\big(S^{-1}\{u^{\alpha}\big[S[L[x]u_2(x,t) + H_2(u)]\big]\}\big)$

$\vdots$

Proceeding in the same manner,the rest of the components $u_n(x,t)$can be completely obtained, and the series solution is thus entirely determined. Finlay we approximate the solution $u(x,t)$by truncated series

$$u(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} u_n\,(x,t)$$

5. APPLICATION

In this section, we discuss the implementation of the proposed method

Example 5.1: For our first example we consider the one dimensional linear inhomogeneous fractional wave equation

$$D_t^\alpha u(x,t) + u_x(x,t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x \, + t\cos x\,, \qquad 0 < \alpha \leq 1,\;\; t > 0 \tag{5.1}$$

subject to the initial condition

$$u(x,0) = 0 \tag{5.2}$$

The exact solution for special case $\alpha = 1$is given by

$$u(x,t) = t \sin x$$

Firstly, applying Sumudu transform on both sided of (5.1) subject to initial condition (5.2), we have

$$S[u(x,t)] = u(x,0) + u^\alpha \left[S\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x \, + t\cos x \right] \right] - u^\alpha \left[S\left(u_x(x,t) \right) \right] \tag{5.3}$$

Operating the inverse Sumudu transform on both sides in (5.3), we have

$$u(x,t) = u(x,0) + S^{-1} \left\{ u^\alpha \left[S\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x \, + t\cos x \right] \right] \right\} - S^{-1}\left\{ u^\alpha \left[S\left(u_x(x,t) \right) \right] \right\} \tag{5.4}$$

Now, applying homotopy perturbation method

$$\sum_{n=0}^{\infty} p^n u_n\,(x,t) = \; u(x,0) + S^{-1} \left\{ u^\alpha \left[S\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x \, + t\cos x \right] \right] \right\} - p\left(S^{-1}\left\{ u^\alpha \left[S\left(u_{nx}(x,t) \right) \right] \right\} \right) \tag{5.5}$$

Equating the corresponding power of on both sides in (5.5), we have

$$p^0 : u_0\,(x,t) = u(x,0) + S^{-1} \left\{ u^\alpha \left[S\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin x \, + t\cos x \right] \right] \right\}$$

$$p^1 : u_1(x,t) = -\left(S^{-1}\left\{ u^\alpha \left[S\left(u_{0x}(x,t) \right) \right] \right\} \right)$$

$$p^2 : u_2(x,t) = -\left(S^{-1}\left\{ u^\alpha \left[S\left(u_{1x}(x,t) \right) \right] \right\} \right)$$

$$p^3 : u_3(x,t) = -\left(S^{-1}\left\{ u^\alpha \left[S\left(u_{3x}(x,t) \right) \right] \right\} \right)$$

$$\vdots$$

Finally, we get

$$u_0\,(x,t) = 0 + S^{-1}\{ u^\alpha [u^{1-\alpha} \sin x \, + u \cos x] \}$$

$$= t \sin x \quad + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos x$$

$$u_1(x,t) = -\left(S^{-1} \left\{ u^\alpha \left[S\left(t\cos x - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \sin x \right) \right] \right\} \right)$$

$$= -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos x + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin x$$

$$u_2(x,t) = -\left(S^{-1}\left\{u^\alpha\left[S\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\sin x + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\cos x\right)\right]\right\}\right)$$

$$= -\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin x - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos x$$

$$u_3(x,t) = -\left(S^{-1}\left\{u^\alpha\left[S\left(-\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\cos x - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\sin x\right)\right]\right\}\right)$$

$$= \frac{-t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos x + \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\sin x$$

$$\vdots$$

$$u(x,t) = t\sin x + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos x - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos x + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin x$$

$$-\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin x - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos x + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos x + \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\sin x$$

$$+\cdots\cdots\cdots$$

For special case $\alpha = 1$ we get

$$u(x,t) = t\sin x$$

which is the exact solution for given problem (5.1) for $\alpha = 1$.

Example 5.2: We consider the following one dimensional reaction Burgers equation:

$$D_t^\alpha u(x,t) + u_x(x,t) - u_{xx}(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, \qquad 0 < \alpha \le 1, \;\; t > 0 \tag{5.6}$$

subject to the initial condition

$$u(x,0) = x^2 \tag{5.7}$$

The exact solution for special case $\alpha = 1$is given by

$$u(x,t) = x^2 + t^2$$

Firstly, applying Sumudu transform on both sided of (5.6) subject to initial condition (5.7), we have

$$S[u(x,t)] = u(x,0) - u^\alpha\left[S\left(u_x(x,t) - u_{xx}(x,t)\right)\right] + u^\alpha\left[S\left[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right]\right] \tag{5.8}$$

Operating the inverse Sumudu transform on both sides in (5.8), we have

$$u(x,t) = u(x,0) + S^{-1}\left\{u^\alpha\left[S\left[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right]\right]\right\} - S^{-1}\left\{u^\alpha\left[S\left(u_x(x,t) - u_{xx}(x,t)\right)\right]\right\} \tag{5.9}$$

Now, applying homotopy perturbation method

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) + S^{-1}\left\{u^\alpha\left[S\left[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right]\right]\right\}$$

$$-p\left(S^{-1}\left\{u^\alpha\left[S\left(u_{nx}(x,t) - u_{nxx}(x,t)\right)\right]\right\}\right) \qquad (5.10)$$

Equating the corresponding power of on both sides in (5.10), we have

$$p^0: u_0(x,t) = u(x,0) + S^{-1}\left\{u^\alpha\left[S\left[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right]\right]\right\}$$

$$p^1: u_1(x,t) = -\left(S^{-1}\left\{u^\alpha\left[S\left(u_{0x}(x,t) - u_{0xx}(x,t)\right)\right]\right\}\right)$$

$$p^2: u_2(x,t) = -\left(S^{-1}\left\{u^\alpha\left[S\left(u_{1x}(x,t) - u_{1xx}(x,t)\right)\right]\right\}\right)$$

$$p^3: u_3(x,t) = -\left(S^{-1}\left\{u^\alpha\left[S\left(u_{2x}(x,t) - u_{2xx}(x,t)\right)\right]\right\}\right)$$

$$\vdots$$

Finally, we get

$$u_0(x,t) = x^2 + t^2 + (2x-2)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$u_1(x,t) = -\left(S^{-1}\left\{u^\alpha\left[S\left(2x - 2 + \frac{2t^\alpha}{\Gamma(\alpha+1)}\right)\right]\right\}\right)$$

$$= -(2x-2)\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^\alpha}{\Gamma(\alpha+1)}$$

$$u_2(x,t) = -\left(S^{-1}\left\{u^\alpha\left[S\left(-\frac{2t^\alpha}{\Gamma(\alpha+1)}\right)\right]\right\}\right)$$

$$= \frac{2t^\alpha}{\Gamma(\alpha+1)}$$

$$u_3(x,t) = -\left(S^{-1}\{u^\alpha[S(0)]\}\right)$$

$$= 0$$

$$\vdots$$

$$u_n(x,t) = 0$$

$$u(x,t) = x^2 + t^2 + (2x-2)\frac{t^\alpha}{\Gamma(\alpha+1)} - (2x-2)\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^\alpha}{\Gamma(\alpha+1)} + 0 + \cdots + 0$$

$$u(x,t) = x^2 + t^2$$

which is the exact solution for given problem (5.6) for $\alpha = 1$.

Example 5.3: Consider the following fourth-order fractional singular partial differential equation

$$D_t^\alpha u(x,t) + \left(\frac{x}{\sin x} - 1\right)u_{xxxx}(x,t) = 0, \quad 0 < \alpha \le 2, 0 < x < 1, \ t > 0 \qquad (5.11)$$

Subject to the initial conditions

$$\begin{cases} u(x,0) = x - \sin x\,, \;\; 0 < x < 1 \\ u_t(x,0) = -(x - \sin x), \;\; 0 < x < 1 \end{cases} \tag{5.12}$$

And boundary conditions

$$\begin{cases} u(0,t) = 0, \;\; u(1,t) = e^{-t}(1 - \sin 1), \;\; t > 0 \\ u_{xx}(0,t) = 0, \; u_{xx}(1,t) = e^{-t} \sin 1 \quad t > 1 \end{cases} \tag{5.13}$$

The exact solution for special case $\alpha = 2$is given by

$$u(x,t) = (x - \sin x\,)e^{-t}$$

Firstly, applying Sumudu transform on both sided of (5.11) subject to initial condition (5.12), we have

$$S[u(x,t)] = u(x,0) + u_t(x,0) - u^{\alpha}\left[S\left(\left(\frac{x}{\sin x} - 1\right)u_{xxxx}(x,t)\right)\right] \tag{5.14}$$

Operating the inverse Sumudu transform on both sides in (5.14), we have

$$u(x,t) = u(x,0) + \;\; u_t(x,0) - S^{-1}\left\{u^{\alpha}\left[S\left(\left(\frac{x}{\sin x} - 1\right)u_{xxxx}(x,t)\right)\right]\right\} \tag{5.15}$$

Now, applying homotopy perturbation method

$$\sum_{n=0}^{\infty} p^n u_n\,(x,t) = \; u(x,t) + \; u_t(x,0) - p\left(S^{-1}\left\{u^{\alpha}\left[S\left(\left(\frac{x}{\sin x} - 1\right)u_{nxxxx}(x,t)\right)\right]\right\}\right) \tag{5.16}$$

Equating the corresponding power of on both sides in (5.16), we have

$$p^0 : u_0\,(x,t) = u(x,0) + \quad u_t(x,0)$$

$$p^1 : u_1(x,t) = -\left(S^{-1}\left\{u^{\alpha}\left[S\left(\left(\frac{x}{\sin x} - 1\right)u_{0xxxx}(x,t)\right)\right]\right\}\right)$$

$$p^2 : u_2(x,t) = -\left(S^{-1}\left\{u^{\alpha}\left[S\left(\left(\frac{x}{\sin x} - 1\right)u_{1xxxx}(x,t)\right)\right]\right\}\right)$$

$$\vdots$$

Finally, we get

$$u_0\,(x,t) = (x - \sin x) - (x - \sin x)t$$

$$u_1(x,t) = -(S^{-1}\{u^{\alpha}[S(-(x - \sin x) - (x - \sin x)t)]\})$$

$$= (x - \sin x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - (x - \sin x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$u_2(x,t) = -\left(S^{-1}\left\{u^{\alpha}\left[S\left(-(x - \sin x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - (x - \sin x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)\right]\right\}\right)$$

$$= (x - \sin x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - (x - \sin x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

$\vdots$

$$u(x,t) = (x - \sin x)\left[1 - t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots + \cdots\cdots\cdots\right]$$

For special case $\alpha = 2$ we get

$$u(x,t) = (x - \sin x)\left[1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \cdots\cdots\cdots\right] u(x,t) = (x - \sin x)\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}$$

$$= (x - \sin x)e^{-t}$$

which is the exact solution for given problem (5.11) for $\alpha = 2$.

Example 5.4: Consider the following fourth-order fractional singular partial differential equation

$$D_t^{\alpha}u(x,t) + \left(\frac{1}{x} + \frac{x^4}{120}\right)u_{xxxx}(x,t) = 0, \quad 0 < \alpha \le 2, \quad \frac{1}{2} < x < 1, \ \ t > 0 \qquad (5.17)$$

Subject to the initial conditions

$$\begin{cases} u(x,0) = 0, \ \ \frac{1}{2} < x < 1 \\ u_t(x,0) = 1 + \frac{x^5}{120}, \ \ 0 < x < 1 \end{cases} \qquad (5.18)$$

and boundary conditions

$$\begin{cases} u(1/2,t) = 0, \qquad u(1,t) = e^{-t}(1 - \sin 1), \ \ t > 0 \\ u_{xx}(1/2,t) = \frac{1}{6}(\frac{1}{2})^3 \sin t, \qquad u_{xx}(1,t) = \frac{1}{6}\sin t \quad t > 0 \end{cases} \qquad (5.19)$$

The exact solution for special case $\alpha = 2$is given by

$$u(x,t) = \left(1 + \frac{x^5}{120}\right)\sin t$$

Firstly, applying Sumudu transform on both sided of (5.17) subject to initial condition (5.18), we have

$$S[u(x,t)] = u(x,0) + \ u_t(x,0) - u^{\alpha}\left[S\left(\left(\frac{1}{x} + \frac{x^4}{120}\right)u_{xxxx}(x,t)\right)\right] \qquad (5.20)$$

Operating the inverse Sumudu transform on both sides in (17), we have

$$u(x,t) = u(x,0) + \ u_t(x,0) - S^{-1}\left\{u^{\alpha}\left[S\left(\left(x + \frac{x^4}{120}\right)u_{xxxx}(x,t)\right)\right]\right\} \qquad (5.21)$$

Now, applying homotopy perturbation method

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) + u_t(x,0) - p\left(S^{-1}\left\{u^{\alpha}\left[S\left(\left(\frac{1}{x}+\frac{x^4}{120}\right)u_{nxxxx}(x,t)\right)\right]\right\}\right) \quad (5.22)$$

Equating the corresponding power of on both sides in (5.22), we have

$$p^0: u_0(x,t) = u(x,0) + \quad u_t(x,0)$$

$$p^1: u_1(x,t) = -\left(S^{-1}\left\{u^{\alpha}\left[S\left(\left(\frac{1}{x}+\frac{x^4}{120}\right)u_{0xxxx}(x,t)\right)\right]\right\}\right)\frac{x}{\sin x} - 1$$

$$p^2: u_2(x,t) = -\left(S^{-1}\left\{u^{\alpha}\left[S\left(\left(\frac{1}{x}+\frac{x^4}{120}\right)u_{1xxxx}(x,t)\right)\right]\right\}\right)$$

$$\vdots$$

Finally, we get

$$u_0(x,t) = 0 + \left(1+\frac{x^5}{120}\right)t$$

$$u_1(x,t) = -\left(S^{-1}\left\{u^{\alpha}\left[S\left(\left(\frac{1}{x}+\frac{x^4}{120}\right)xt\right)\right]\right\}\right)$$

$$= -\left(1+\frac{x^5}{120}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$u_2(x,t) = -\left(S^{-1}\left\{u^{\alpha}\left[S\left[\left(\frac{1}{x}+\frac{x^4}{120}\right)\left(-\frac{t^{\alpha}}{\Gamma(\alpha+1)}x\right)\right]\right]\right\}\right)$$

$$= \left(1+\frac{x^5}{120}\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

$$\vdots$$

$$u(x,t) = \left(1+\frac{x^5}{120}\right)\left[t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots\cdots\right]$$

$$u(x,t) = \left(1+\frac{x^5}{120}\right)\sum_{k=0}^{n}\frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)}$$

For special case $\alpha = 2$ we get

$$u(x,t) = \left(1+\frac{x^5}{120}\right)\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\cdots\cdots\right]$$

$$u(x,t) = \left(1+\frac{x^5}{120}\right)\sin t$$

which is the exact solution for given problem (5.19) for $\alpha = 2$.

6.CONCLUSION:

In present paper, (HPSTM) for finding exact solutions of some fractional partial differential equations by employing integral transform coupled to homotopy perturbation method is proposed. The coupling is based on Caputo fractional derivative definition solely owing to its flexibility. On the same side, the method is applied to some test examples yielding exact solutions. Thus the presented method can be used in solving various fractional models as convergent series with easily computable components.

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